

# Annulus amplitude of FZZT branes revisited

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## Abstract

We revisit the annulus amplitude of FZZT branes with general matter sectors  $(r, s)$  using the recent development of matrix model and minimal Liouville gravity. Following the boundary description of the 1-matrix model and bulk resonance transformation between primary operators we find the consistency of the brane decomposition into  $(1, 1)$ -branes. We also investigate the corresponding results obtained directly from the minimal Liouville gravity and demonstrate the perfect agreement with the matrix results.

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# 1 Introduction

The quantum gravity in two space-time dimensions can be described in terms of Liouville gravity [1] and its non-perturbative effect of interaction with matter is reliably investigated if conformal symmetry is maintained. The interaction with minimal matter is studied in the name of minimal Liouville gravity (MLG). The minimal gravity is simple because the number of primary fields is finite and the exact correlation numbers (expectation values of coordinate-integrated form of correlation functions) can be obtained.

MLG is also studied in the context of matrix models.  $(2, 2p + 1)$  minimal Liouville gravity is described by the hermitian 1-matrix model (sometimes called as  $p$ -critical model) [2] and by 2-matrix model [3] the minimal  $(q_1, q_2)$  unitary theory (with  $q_1 < q_2$  co-prime). The comparison of the matrix model with MLG is a non-trivial task [4]. Nonetheless, the parameter dependence of MLG is conjectured on the fluctuation sphere and its exact form is provided for the case of the Lee-Yang matter ( $p = 2$ ) [5]. For  $p \geq 3$  one needs to consider the resonance between primary operators. The exact bulk resonance transformation (BZ transformation) is conjectured for the  $p$ -critical case [6] and is tested up to some of five-point correlations [7].

When one considers boundaries in MLG, one needs to specify the boundary condition, which is represented by D-branes. Possible D-branes, FZZT in MLG is discussed in [8, 9]. The boundary state is given by the tensor product of that of Liouville theory and that of minimal model and is specified by the continuous boundary parameter  $s$  and by the two integers  $(k \leq q_1, \ell \leq q_2)$  in the  $(q_1, q_2)$  MLG. It is conjectured that not all of these states are independent but is argued that general boundary states coming from  $(k, \ell)$  states are linear combination of  $(1, 1)$ -brane. Specifically,

$$|s; (k, \ell)\rangle = \sum_{m' = -(k-1), 2}^{k-1} \sum_{n' = -(l-1), 2}^{l-1} |s + im' \frac{1}{b} + in'b; 1, 1\rangle \quad (1.1)$$

where  $b = \sqrt{\frac{q_1}{q_2}}$ . This relation is checked at the ground ring level in [10, 11]. With this conjecture, most of the interest is centered on  $(1, 1)$  brane whose matrix object is associated with the macroscopic loop operator

$$\left\langle \text{tr} \frac{1}{u_0 z - M} \right\rangle \quad (1.2)$$

where  $u_0$  is proportional to the square root of the bulk cosmological constant with KP scaling 1 (We do not elaborate on this fine tuning at the critical limit further; one may refer to *e.g.* [12, 13]) and  $z$  is related to the continuous boundary cosmological constant parameter  $s$

$$z = \cosh(\pi b s). \quad (1.3)$$

The disk partition function of the matrix model is given as  $\mathcal{Z} = \langle \text{tr} \ln(z - M) \rangle$ .

For other branes with general matter sector little study has been done until recently. Indeed, the disk partition function of  $p$ -critical model with BC  $(s, (1, m))$  is given as [12]

$$\mathcal{Z}_{\text{disk}}(s; (1, m)) = \langle \text{tr} \log F_m(z, M) \rangle, \quad F_m(z, M) = \prod_{k=-(m-1):2}^{m-1} (u_0 z_k - M) \quad (1.4)$$

where  $z_k = \cosh(\pi b s_k)$  with  $s_k = s + ibk$ . This proposal is obviously consistent with Eq. (1.1) with  $k = 1$  since we are dealing with  $(2, 2p+1)$  case out of general  $(q_1, q_2)$ : The partition function is simply addition of that of  $(1, 1)$ -boundary with cosmological constant parameter shifted by suitable imaginary value, consistent with the brane decomposition of MLG. In addition, it is obvious how to generalize the above proposal Eq. (1.4) to  $(q_1, q_2)$ -model by considering 2-matrix model [13]. Nontrivial tests for this proposal were carried out in [12, 13] at the disk level. It is confirmed that the disk one and two-point correlations in the matrix model reproduce the known results of Liouville theory[8].

This idea of decomposition of the branes are very intuitive and the idea should go beyond the disk boundary. Given the prescription of Eq. (1.4), it is straightforward to work out the corresponding annulus amplitude in the matrix model and compare the results of the matrix model with the corresponding MLG, which is the main theme of this paper. In this paper we carefully work out the annulus amplitude and find the perfect agreement between the matrix model proposal and the MLG computation, thereby confirming the proposal of [12, 13] at annulus geometry.

Incidentally, this solves the confusions recently raised on the MLG results [14, 15]. The annulus amplitude is evaluated in [16] using the boundary Louville field theory and lattice height model of  $A_{q_1-1}$  series [17]. For example, for the  $(1, 1)$ -boundary, the annulus amplitude is given as

$$\mathcal{Z}(s, 1|s', 1) = \int_{-\infty}^{\infty} \frac{d\nu}{\nu} \frac{\cos(\pi s \nu) \cos(\pi s' \nu) \sinh(\pi(q_1 - 1)\nu/b)}{\sinh(\pi \nu/b) \sinh(\pi q_1 \nu/b)}. \quad (1.5)$$

This shows a subtle point since the amplitude is to be regulated to avoid the divergence at  $\nu = 0$ . The subtlety raises questions about the universal form of the annulus amplitude [14, 15] and about the decomposition into  $(1, 1)$ -branes [15].

The content of the paper is as follows. In section 2, we check the annulus amplitude in the matrix model. For  $(1, 1)$  boundary, we use the boundary 1-matrix model [12] and evaluate the amplitude using the filling fraction representation. In this way, the universal contribution of the amplitude is identified. And it is straight-forward to write down the annulus amplitude for branes with general matter sectors. In section 3, we revisit the Liouville theory computation of the minimal gravity obtained in [15]. After using the summation formula to get the compact form of the amplitude, one can explicitly demonstrate that the result reduces to the formula (1.5). In addition, we present the annulus amplitude for the general boundaries and find the result consistent with the brane-decomposition. Section 4 is the conclusion and discussion where bulk correlation in the annulus is presented for the  $p$ -critical model using the BZ transformation. In appendix, one can find detailed calculations.

## 2 Annulus amplitude in the matrix model

The  $p$ -critical model ( $q_1 = 2, q_2 = 2p + 1$ ) is described in terms of one-matrix model. Even though the  $p$ -critical model is non-unitary series, the finite number of primary operators produces many properties sharing with the unitary series ( $q_1 \geq 3$ ). Thus, we start with the one-matrix model for simplicity. The annulus amplitude is obtained from the two-loop correlation

$$W_2(z^{(1)}, z^{(2)}) = \left\langle \text{tr} \left( \frac{1}{u_0 z^{(1)} - M} \right) \text{tr} \left( \frac{1}{u_0 z^{(2)} - M} \right) \right\rangle \quad (2.1)$$

where  $z^{(i)} = \cosh(\pi b s^{(i)})$ . Explicit evaluation shows [18, 19, 20]

$$W_2(z^{(1)}, z^{(2)}) = \frac{\partial}{\partial z^{(1)}} \frac{\partial}{\partial z^{(2)}} \log \left( \frac{\zeta^{(1)} - \zeta^{(2)}}{z^{(1)} - z^{(2)}} \right) = -\frac{\partial}{\partial z^{(1)}} \frac{\partial}{\partial z^{(2)}} \log(\zeta^{(1)} + \zeta^{(2)}) \quad (2.2)$$

where  $\zeta^{(i)} = \cosh(\pi b s^{(i)}/2)$  covers the double-sheet parameter space. The definition of the annulus amplitude  $W(z^{(1)}, z^{(2)}) = \frac{\partial}{\partial z^{(1)}} \frac{\partial}{\partial z^{(2)}} \mathcal{Z}(z^{(1)}, z^{(2)})$  results in

$$\mathcal{Z}(z^{(1)}, z^{(2)}) = -\log(\zeta^{(1)} + \zeta^{(2)}) + f_1(z^{(1)}) + f_2(z^{(2)}) \quad (2.3)$$

where  $f_i$ 's are function of  $z^{(1)}$  or  $z^{(2)}$  only. The result is consistent with Eq. (1.5).

The remaining subtle point is the regularization dependency and the universal behavior of the annulus amplitude [14, 15]. To clarify these, we provide another useful and simple formula for the annulus amplitude in terms of filling fraction integral representation and BZ resonance in [6]. The nice feature of this representation is that one can pin-point the universal contribution precisely. We may put the two-loop correlation (2.1) using the Laplace transformation

$$W(z^{(1)}, z^{(2)}) = \int_{\ell^{(1)}, \ell^{(2)} \geq 0} e^{-(\ell^{(1)} z^{(1)} + \ell^{(2)} z^{(2)})} C(\ell^{(1)}, \ell^{(2)}) \quad (2.4)$$

where  $C(\ell^{(1)}, \ell^{(2)})$  is interpreted as the annulus amplitude with fixed lengths

$$C(\ell^{(1)}, \ell^{(2)}) = \int_1^\infty dx \int_0^1 d\tilde{y} \langle x | e^{\ell^{(1)}(u_0 d^2 - u)} | \tilde{y} \rangle \langle \tilde{y} | e^{\ell^{(2)}(u_0 d^2 - u)} | x \rangle \quad (2.5)$$

where  $d$  denotes the differential operator with respect to the filling fraction  $x, \tilde{y}$  of the matrix eigenvalues. Note that the integration range does not overlap except  $x = \tilde{y} \neq 0$  so that  $\langle \tilde{y} | x \rangle = 0$ . To proceed, we change the variables  $x \rightarrow 1 - x$  and  $\tilde{y} \rightarrow 1 - \tilde{y}$  and use  $x + Q_p(u) = 0$  which sets  $u$  as a certain function of  $x$  through BZ transformation. (Note that the string equation is given as  $Q_p(u_*) = 0$ ). This identification translates the matrix result (kdV frame) into the field theory one (CFT frame).  $Q_p(u)$  is given in terms of the Legendre polynomial  $L_p(\xi)$  with  $\xi = u/u_0$  [6] (in the absence of the bulk couplings),

$$Q_p(u) = \frac{L_{p+1}(\xi) - L_{p-1}(\xi)}{2p + 1}. \quad (2.6)$$

One may evaluate (2.5) with the help of the momentum integration

$$C(\ell^{(1)}, \ell^{(2)}) = \frac{1}{\sqrt{\ell^{(1)} \ell^{(2)}}} \int_{-\infty}^0 dx \int_0^1 d\tilde{y} e^{-(x-\tilde{y})^2(\ell^{(1)} + \ell^{(2)})/(\ell^{(1)} \ell^{(2)})} e^{-\ell^{(1)} u(x) - \ell^{(2)} u(\tilde{y})}. \quad (2.7)$$

$C(\ell^{(1)}, \ell^{(2)})$  is proportional to  $\sqrt{\ell^{(1)} \ell^{(2)}}/(\ell^{(1)} + \ell^{(2)})$  as  $\ell^{(1)}, \ell^{(2)} \rightarrow 0$ . After integration over the length variables of the annulus amplitude (2.4) one has the universal form

$$\mathcal{F}(z^{(1)}, z^{(2)}) = \int_{x, \tilde{y}} \frac{e^{-|x-\tilde{y}|R_0}}{(x-\tilde{y})^2} + f_1(z^{(1)}) + f_2(z^{(2)}) \quad (2.8)$$

where  $\int_{x, \tilde{y}} \equiv \int_{-\infty}^0 dx \int_0^\infty d\tilde{y}$  and  $R_0 = \sqrt{2}(\zeta^{(1)} + \zeta^{(2)})$ . We distinguish  $\mathcal{F}(z^{(1)}, z^{(2)})$  from  $\mathcal{Z}(z^{(1)}, z^{(2)})$  for later use. In addition, we change the integration limit of  $\tilde{y}$  from 1 to  $\infty$

since this addition does not change the universal part because the universal contribution comes from the region where the string equation  $Q_p(u_*) = 0$  is satisfied (at  $x = \tilde{y} = 0$ ), whose solution is  $u_* = u_0$  (See details in Appendix A).

It is noted that the integral is not convergent at  $x = \tilde{y} = 0$ . To make the integration finite, one may choose the integration constants  $f_i$ 's so that the integral  $\mathcal{F}(0, 0) = 0$

$$\mathcal{F}(z^{(1)}, z^{(2)}) = -\log \left( \frac{\zeta^{(1)} + \zeta^{(2)}}{2} \right) \quad (2.9)$$

which is consistent with Eq. (2.3). One may wonder if one can remove  $\log(z^{(1)} - z^{(2)})$  by a suitable regularization. However, it is obvious that that choice is impossible.

The annulus amplitude with boundaries  $(s^{(1)}, (1, m)), (s^{(2)}, (1, \ell))$  is proposed in [12]

$$\mathcal{Z}_{\text{ann}}(s^{(1)}, (1, m)|s^{(2)}, (1, \ell)) = \langle \text{tr} \log F_m(z^{(1)}, M) \text{tr} \log F_\ell(z^{(2)}, M) \rangle_c \quad (2.10)$$

where  $\langle \rangle_c$  stands for the connected part of the partition function. According to this, the amplitude is consistent with the decomposition of the  $(1, 1)$ -branes [14]

$$\mathcal{Z}_{\text{ann}}(s^{(1)}, (1, m)|s^{(2)}, (1, \ell)) = \sum_{m' = -(m-1); 2}^{m-1} \sum_{\ell' = -(\ell-1); 2}^{\ell-1} \mathcal{Z}_{\text{ann}}(s_{m'}^{(1)}, (1, 1)|s_{\ell'}^{(2)}, (1, 1)). \quad (2.11)$$

Similar decomposition for the boundary 2-matrix model can be checked [13] by extending (2.10) into 2-matrix version

$$\begin{aligned} & \mathcal{Z}_{\text{ann}}(s^{(1)}, (r, m)|s^{(2)}, (s, \ell)) \\ &= \sum_{s' = -(s-1); 2}^{s-1} \sum_{m' = -(m-1); 2}^{m-1} \sum_{r' = -(r-1); 2}^{r-1} \sum_{\ell' = -(\ell-1); 2}^{\ell-1} \mathcal{Z}_{\text{ann}}(s_{s'}^{(1)}, (1, 1)|s_{r', \ell'}^{(2)}, (1, 1)) \end{aligned} \quad (2.12)$$

where  $s_{r, m}^{(i)} = s^{(i)} + ir/b + imb$ .

### 3 Annulus amplitude in minimal Liouville gravity

Now let us investigate  $(q_1, q_2)$ -MLG: The annulus amplitude is considered in [15, 23]. For  $(1, 1)$ -boundary one has<sup>1</sup>

$$\mathcal{Z}((1, 1), s|(1, 1), s') = -\frac{1}{2q_1q_2} \int_{-\infty}^{\infty} \frac{d\eta \cos(\sqrt{q_1q_2}\eta s) \cos(\sqrt{q_1q_2}\eta s') \sinh \eta}{\eta \sinh(q_1\eta) \sinh(q_2\eta)} F_{1,1}(i\eta) \quad (3.1)$$

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<sup>1</sup>Normalization of the Liouville part is taken so that  $2\sqrt{2}\pi^2$  is absent and - sign correction is done in (3.10) of [15]

where

$$F_{1,1}(z) = \sum_{\alpha=1}^{q_1-1} \sum_{\beta=-(q_2-1)}^{q_2-1} \frac{\sin(\pi t/q_1) \sin(\pi t/q_2)}{\cos(\pi t/(q_1 q_2)) - \cos z} \quad (3.2)$$

with  $t = \alpha q_2 + \beta q_1$ . After summation,  $F_{1,1}(z)$  is given in a compact form<sup>2</sup>

$$F_{1,1}(z) = -2q_1 q_2 \frac{\sin(z(q_1 - 1)q_2) \sin(zq_1)}{\sin(zq_1 q_2) \sin(z)}. \quad (3.3)$$

This shows that the annulus amplitude reproduces<sup>3</sup> exactly the same result (1.5) ( $\eta \rightarrow \pi\nu/\sqrt{q_1 q_2}$ ). Thus, one concludes that the annulus amplitude for  $(q_1, q_2)$ -minimal gravity will be in the form [14]

$$\mathcal{Z}(z^{(1)}, z^{(2)}) = \log \left( \frac{\zeta_{q_1 q_2}^{(1)} - \zeta_{q_1 q_2}^{(2)}}{T_{q_1}(\zeta_{q_1 q_2}^{(1)}) - T_{q_1}(\zeta_{q_1 q_2}^{(2)})} \right) \quad (3.4)$$

where  $\zeta_{q_1 q_2}^{(i)} = \cosh(\pi s^{(i)}/\sqrt{q_1 q_2})$ .

To see the general boundary amplitude, let consider  $q_1 = 2$  case first. According to the result of [15], the numerator term  $\sin(\pi t/q_2)$  in (3.2) is modified into

$$\begin{aligned} F_{m,\ell}(z) &= \sum_{\beta=-(q_2-1)}^{q_2-1} \frac{\sin(\pi t/q_1) \left\{ \sin(\pi t m/q_2) \sin(\pi t \ell/q_2) / \sin(\pi t/q_2) \right\}}{\cos(\pi t/(q_1 q_2)) - \cos z} \\ &= -2q_1 q_2 \frac{\sin(z(q_1 - 1)q_2) \sin(zq_1 \ell)}{\sin(zq_1 q_2) \sin(z)}. \end{aligned} \quad (3.5)$$

where  $t = q_2 + \beta q_1$  and without loss of generality  $1 \leq \ell, m \leq (q_2 - 1)/2$ . Putting this into the annulus amplitude (3.1) one has the matrix result (2.11) (Note that the numerator  $\cos(\sqrt{q_1 q_2} \eta s) \sinh(\eta q_1 m) / \sinh(\eta q_1)$  is decomposed into the sum of  $\cos(\sqrt{q_1 q_2} \eta s_{m'})$ ).

It is a simple matter to confirm the unitary series  $q_1 \geq 3$ , using the formula (C.3) and (C.4) in App. C that the amplitude indeed satisfies the decomposition (2.12)

## 4 Conclusion and discussion

We provide the explicit form of the annulus amplitude for  $(1, 1)$  boundary in two different approaches, one using the boundary matrix model and the other using the minimal

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<sup>2</sup>One can use the same trick given in [14] using the pole structure and the large imaginary behavior in  $z$  for  $q_1 < q_2$ . See the details in App. C.

<sup>3</sup>It seems that (3.16) in [15] does not go with this observation.

Liouville gravity. The universal part of the matrix model agrees with the one given from the Liouville gravity side, even though one needs to regularize the annulus amplitude. It is noted that the universal contribution of the annulus amplitude of the 1-matrix model (the  $p$ -critical model,  $q_1 = 2$  and  $q_2 = 2p + 1$ ) is given as  $\log(\zeta^{(1)} + \zeta^{(2)})$  rather than  $\log(\zeta^{(1)} - \zeta^{(2)})$  [14]. On the other hand, the annulus amplitude of the general boundary is decomposed into the sum of  $(1, 1)$  boundaries as proposed in [14], with the Liouville boundary parameters are (imaginary) shifted.

After this convincing evidence for the annulus amplitude from the matrix side, one may calculate bulk correlation of the  $p$ -critical model in the annulus using the formula in (2.3) if one applies the BZ transformation in the presence of the bulk source

$$Q_p(u, \lambda_k) = \frac{L_{p+1}(\xi) - L_{p-1}(\xi)}{2p+1} + \lambda_k L_{p-k} + O(\lambda^2) \quad (4.1)$$

where  $\lambda_k$  is the source to the dressed bulk operator  $O_k = \int_M e^{2b\alpha_k\varphi} \Phi_k$  with  $\alpha_k = (k+1)/2$  ( $k = 2, \dots, p$ ).  $\varphi$  is the Liouville field and  $\Phi_k$  represents the matter field with  $\Phi_1 = I$ . The bulk correlation is defined as  $\mathcal{F}(z^{(1)}, z^{(2)}; O_k) \equiv -\frac{\partial}{\partial \lambda_k} \mathcal{F}(z^{(1)}, z^{(2)}) \Big|_{\lambda_k=0}$ . Using the properties;  $x(\xi, \lambda_k) = -Q_p(u, \lambda_k)$  so that  $x \equiv x(\xi) = -Q_p(u, 0)$  with the conditions  $x(\xi=0) = 0$  and

$$\frac{\partial dx(\xi, \lambda_k)}{\partial \lambda_k} \Big|_{\lambda=0} = dx \frac{L'_{p-k}(\xi)}{L_p(\xi)}, \quad \frac{\partial x(\xi, \lambda_k)}{\partial \lambda_k} \Big|_{\lambda=0} = -L_{p-k}(\xi) \quad (4.2)$$

one has finite result with KP scaling factor  $u_0^{-2\alpha_k}$  (noting that the subtracted term in (2.8) or (A.4) has no  $\lambda_k$ -dependence)

$$\mathcal{F}(z^{(1)}, z^{(2)}; O_k) = - \int_{x, \tilde{y}} \frac{e^{-|x-\tilde{y}|R_0}}{(x-\tilde{y})^2} g(x, \tilde{y}), \quad (4.3)$$

where  $g = g_e + g_o$

$$\begin{aligned} g_e(x, \tilde{y}) &= g_e(\tilde{y}, x) = \frac{L'_{p-k}(\xi)}{L_p(\xi)} + \frac{L'_{p-k}(\tilde{\xi})}{L_p(\tilde{\xi})} + 2 \frac{L_{p-k}(\tilde{\xi}) - L_{p-k}(\xi)}{\tilde{y} - x} \\ g_o(x, \tilde{y}) &= -g_o(\tilde{y}, x) = (L_{p-k}(\tilde{\xi}) - L_{p-k}(\xi)) R_0. \end{aligned} \quad (4.4)$$

The integration is not simple to carry out. By noting that rescaling  $x$  and  $\tilde{y}$  by  $1/R_0$  is broken in  $g$ , one may evaluate  $\mathcal{F}(z^{(1)}, z^{(2)}; O_k)$  in  $1/R_0$  expansion. In fact, the  $x$  and  $\tilde{y}$  symmetry enforces the odd power of  $1/R_0$  to vanish and has the form of expansion  $F(z^{(1)}, z^{(2)}; O_k) = \sum f_n R_0^{-2n}$  where  $f_n$  is a constant which depends only on  $p$  and  $k$ . Explicit calculation shows that  $f_0 = (p+1-k)(p-k)/2$  and  $f_2 = 0$  (see Appendix C).

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## Appendix

In this Appendix, we provide details of the calculation needed in the text.

### A Evaluation of the annulus amplitude

To find  $\mathcal{F}(z^{(1)}, z^{(2)})$  in (2.8), we first integrate (2.4) over  $\ell^{(i)}$ 's

$$W(z^{(1)}, z^{(2)}) = \int_{-\infty}^0 dx \int_0^\epsilon d\tilde{y} \int_{p_1, p_2} \frac{e^{i(p_1 - p_2)(x - \tilde{y})}}{(u_0 p_1^2 + u(x) + z^{(1)})(u_0 p_2^2 + u(\tilde{y}) + z^{(2)})}. \quad (\text{A.1})$$

The annulus amplitude is given as  $\mathcal{F}(z^{(1)}, z^{(2)}) = \mathcal{F}_0(z^{(1)}, z^{(2)}) + f_1(z^{(1)}) + f_2(z^{(2)})$  where  $f_i$ 's are integration constants and

$$\mathcal{F}_0(z^{(1)}, z^{(2)}) = \int_{x, \tilde{y}} \int_{p_1, p_2} e^{i(p_1 - p_2)(x - \tilde{y})} \log(u_0 p_1^2 + u(x) + z^{(1)}) \log(u_0 p_2^2 + u(\tilde{y}) + z^{(2)}) \quad (\text{A.2})$$

with the shorthand notation<sup>4</sup>,  $\int_{x, \tilde{y}} \equiv \int_{-\infty}^0 dx \int_0^\epsilon d\tilde{y}$ . After integration by part of the momenta one has

$$\begin{aligned} \mathcal{F}_0(z^{(1)}, z^{(2)}) &= \int_{x, \tilde{y}} \int_{p_1, p_2} \frac{\log(u_0 p_1^2 + u(x) + z^{(1)}) \log(u_0 p_2^2 + u(\tilde{y}) + z^{(2)})}{(x - \tilde{y})^2} \frac{\partial}{\partial p_1} \frac{\partial}{\partial p_2} e^{i(p_1 - p_2)(x - \tilde{y})} \\ &= \int_{x, \tilde{y}} \frac{1}{(x - \tilde{y})^2} \int_{p_1, p_2} e^{i(p_1 - p_2)(x - \tilde{y})} \left( \frac{1}{p_1 + i\sqrt{\xi(x) + z^{(1)}}} + \frac{1}{p_1 - i\sqrt{\xi(x) + z^{(1)}}} \right) \\ &\quad \times \left( \frac{1}{p_2 + i\sqrt{\xi(\tilde{y}) + z^{(2)}}} + \frac{1}{p_2 - i\sqrt{\xi(\tilde{y}) + z^{(2)}}} \right) \end{aligned}$$

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<sup>4</sup>The integration range is originally  $0 < \tilde{y} < 1$ . We put  $\epsilon$  in the integration limit for later convenience.

$$= \int_{x, \tilde{y}} \frac{e^{-|x-\tilde{y}|R_{12}(\xi, \tilde{\xi})}}{(x - \tilde{y})^2} \quad (\text{A.3})$$

where  $\xi = u(x)/u_0 \geq 1$ ,  $\tilde{\xi} = u(\tilde{y})/u_0 \leq 1$ , and  $R_{12}(\xi, \tilde{\xi}) = \sqrt{\xi + z^{(1)}} + \sqrt{\tilde{\xi} + z^{(2)}}$  is real and positive.

The integration in (A.3) will give divergent contribution in general when  $x = \tilde{y}$ . One can make the amplitude finite after subtracting this divergence using the integration constants  $f_i(z^{(i)})$ 's, since the divergence is independent of  $z^{(i)}$ 's. Suppose one requires  $F(z^{(1)} = z^{(2)} = 0|R) = 0$ , one has

$$F(z^{(1)}, z^{(2)}|R) = \int_{x, \tilde{y}} \frac{e^{(x-\tilde{y})R_{12}(\xi, \tilde{\xi})}}{(x - \tilde{y})^2} - (R_{12}(\xi, \tilde{\xi}) \rightarrow \tilde{R}_{12}(\xi, \tilde{\xi})) \quad (\text{A.4})$$

where  $\tilde{R}_{12}(\xi, \tilde{\xi}) = R_{12}(\xi, \tilde{\xi})|_{z^{(1)}=z^{(2)}=0}$ .

Let us evaluate the universal contribution of  $F(z_1, z_2|R)$  in (A.4). One may add the contribution  $\tilde{y} > 1$  without affecting the universal part. Note also that at  $x = \tilde{y} = 0$ , the string equation  $Q_p(u) = 0$  has the solution  $\xi = \tilde{\xi} = 1$ . Therefore, the universal contribution can be of the form if one put  $R_{12} \rightarrow R_0 = \sqrt{1 + z^{(1)}} + \sqrt{1 + z^{(2)}}$  and  $\epsilon \rightarrow \infty$

$$\partial_{R_0}^2 F(z_1, z_2|R_0) = \lim_{\epsilon \rightarrow \infty} \int_{x\tilde{y}} e^{-(x+\tilde{y})R_0} = \frac{1}{R_0^2}. \quad (\text{A.5})$$

Integrating over  $R_0$  twice, one has

$$F(z^{(1)}, z^{(2)}|R_0) = -\log \left( \frac{R_0(z^{(1)}, z^{(2)})}{R_0(0, 0)} \right) = -\log \left( \frac{\zeta_1 + \zeta_2}{2} \right) \quad (\text{A.6})$$

by requiring  $F(z^{(1)} = z^{(2)} = 0|R) = 0$ .

## B Evaluation of the bulk-annulus amplitude

The bulk correlation in the annulus in (4.3) is calculated in  $1/R_0$  expansion. First note that  $\xi$  is the function of  $x$  and its explicit form can be found by expanding around  $\xi = \tilde{\xi} = 1$ ,

$$\begin{aligned} x = -Q_p(u)|_{\lambda=0} &= -(\xi - 1) - (\xi - 1)^2 \left( \frac{L'_p(1)}{2} \right) - (\xi - 1)^3 \left( \frac{L_p^{(2)}(1)}{6} \right) + O((\xi - 1)^4), \\ \xi &= 1 - x - x^2 \left( \frac{L'_p(1)}{2} \right) - x^3 \left( \frac{1}{2} (L'_p(1))^2 - \frac{1}{6} L_p^{(2)}(1) \right) + O(x^4). \end{aligned} \quad (\text{B.1})$$

Then  $g(x, \tilde{y})$  in (4.4) is expanded as (with  $m + n = 3$ )

$$g_e(x, \tilde{y}) = g_e^{(0)} + g_e^{(1)}(x + \tilde{y}) + g_e^{(2)}(x^2 + \tilde{y}^2) + g_e^{(1,1)}x\tilde{y} + O(x^m\tilde{y}^n) \quad (\text{B.2})$$

$$g_o(x, \tilde{y}) = (x - \tilde{y})R_0 (g_o^{(0)} + g_o^{(1)}(x + \tilde{y}) + g_o^{(2)}(x^2 + \tilde{y}^2 + x\tilde{y}) + O(x^m\tilde{y}^n)) \quad (\text{B.3})$$

with the coefficients  $g_e^{(i)}$  and  $g_o^{(i)}$  which depend on  $p$  and  $k$  only. Explicit calculation shows that  $g_e^{(0)} = g_o^{(1)} = 0$ . In addition, the term with  $g_e^{(1)}$  vanishes when integration is done due to the exchange symmetry of  $x$  and  $\tilde{y}$ . The term with  $g_o^{(0)} = (p + 1 - k)(p - k)/2$  is  $R_0$  independent when integrated out and the rest terms give

$$\int_{x\tilde{y}} \frac{e^{-|x-\tilde{y}|R_0}}{|x-\tilde{y}|^2} \left( g_e^{(2)}(x^2 + \tilde{y}^2) + g_e^{(1,1)}x\tilde{y} + R_0 g_o^{(2)}(x - \tilde{y})(x^2 + \tilde{y}^2 + x\tilde{y}) \right) = \frac{A}{R_0^2} \quad (\text{B.4})$$

with  $A = \frac{2}{3}g_e^{(2)} - \frac{1}{6}g_e^{(1,1)} - g_o^{(2)} = 0$ .

## C Summation formula

We provide useful summation formula. When  $p$  and  $a$  are integers and  $1 \leq a < p$ , one has

$$\sum_{j=1}^{p-1} \frac{\sin^2(\pi j a/p)}{\cosh(\pi j/p) - \cosh(\pi \xi/p)} = -p \frac{\sin(\pi \xi(1 - a/p)) \sin(\pi \xi a/p)}{\sinh(\pi \xi) \sinh(\pi \xi/p)}. \quad (\text{C.1})$$

One can check that both sides have same poles and residues. In addition, the leading behavior as  $\xi \gg 1$ , the leading behavior is  $e^{-\pi \xi/p}$  with the same coefficient,  $-p$ .

When  $p$  and  $a, b$  are integers and  $1 \leq a, b < p$ , one has [14]

$$\sum_{j=1}^{p-1} \frac{\sin(\pi j a/p) \sin(\pi j b/p)}{\cosh(\pi j/p) - \cosh(\pi \xi/p)} = -p \frac{\sin(\pi \xi(1 - A - B)) \sin(\pi \xi(A - B))}{\sinh(\pi \xi) \sinh(\pi \xi/p)} \quad (\text{C.2})$$

where  $A = (a + b)/(2p)$  and  $B = |a - b|/(2p)$ . This can be obtained from (C.1) by changing the numerator of LHS as two terms of sine squared using the formula  $\sin x \sin y = \sin^2 \frac{x+y}{2} - \sin^2 \frac{x-y}{2}$ .

When  $q_1 < q_2$  are co-prime numbers with integers  $k_1$  and  $k_2$  ( $1 \leq k_1 < q_1$  and  $1 \leq k_2 < q_2$ ), one has

$$\sum_{\alpha=1}^{q_1-1} \sum_{\beta=-(q_2-1)}^{q_2-1} \frac{\sin(\pi t k_1/q_1) \sin(\pi t k_2/q_2)}{\cosh(\pi t/\tau) - \cosh(\pi \xi/\tau)} = -2\tau \frac{\sin(\pi \xi(1 - A - B)) \sin(\pi \xi(A - B))}{\sinh(\pi \xi) \sinh(\pi \xi/\tau)} \quad (\text{C.3})$$

where  $A = k_1/q_1 + k_2/q_2$ ,  $B = |k_1/q_1 - k_2/q_2|$ ,  $t = \alpha q_2 + \beta q_1$  and  $\tau \equiv q_1 q_2$ .

Finally, when  $m$  and  $\ell$  are integers  $1 \leq m, \ell < q$ , one has

$$\frac{\sin(xm/q)}{\sin(x/q)} \frac{\sin(x\ell/q)}{\sin(x/q)} = \sum_k C_k^{m,\ell} \sin(xk/q) / \sin(x/q) \quad (\text{C.4})$$

where  $C_k^{m,\ell}$  is a certain integer and satisfies  $C_k^{m,\ell} = C_k^{q-m, q-\ell}$ .

## References

- [1] A. Polyakov, Phys. Lett. **B103** (1981) 207.
- [2] V. A. Kazakov, Mod. Phys. Lett. **A4** (1989) 2125.
- [3] V. A. Kazakov, Phys. Lett. **A119** (1986) 140.
- [4] G. Moore, N. Seiberg, and M. Staudacher, Nucl. Phys. **B362** (1991) 665.
- [5] Al. Zamolodchikov, “Perturbed Conformal Field Theory on Fluctuating Sphere”,  
arXiv:hep-th/0508044 (2005).
- [6] A.A. Belavin and A. Zamolodchikov, J. Phys. **A42** (2009) 304004.
- [7] G. Tarnopolsky, J.Phys.A44:325401,2011 [arXiv: 0912.4971[hep-th]].
- [8] V. Fateev, A. B. Zamolodchikov and Al. B. Zamolodchikov, “Boundary Liouville  
Field Theory I. Boundary State and Boundary Two-point Function”,  
[arXiv:hep-th/0001012].
- [9] J. Teschner, “Remarks on Liouville theory with boundary”,  
[arXiv:hep-th/0009138]
- [10] N. Seiberg and D. Shih, JHEP **0402** (2004) 021 [arXiv:hep-th/0312170].
- [11] A. Basu and E. J. Martinec, Phys.Rev. D72 (2005) 106007 [arXiv:hep-th/0509142].
- [12] G. Ishiki and C. Rim, Phys. Lett. **B694**, 272 (2010) [arXiv:hep-th/1006.3906];  
J. E. Bourguine, G. Ishiki and C. Rim, Phys. Lett **B698** (2011) 68.  
[arXiv:hep-th/1010.1363]; J. E. Bourguine, G. Ishiki and C. Rim,  
[arXiv:1107.4186];

- [13] J. E. Bourguine, G. Ishiki and C. Rim, JHEP **1012:046** (2010)  
[arXiv:hep-th/1010.1363].
- [14] D. Kutasov, K. Okuyama, J. Park, N. Seiberg and D. Shih, JHEP **0408:026** (2004)  
[arXiv:hep-th/0406030].
- [15] M. Atkin and J. Wheeler, JHEP **1102:084** (2011) [arXiv:1011.5989] .
- [16] E. Martinec, “The Annular Report on Non-Critical String Theory”,  
[arXiv:hep-th/0305148].
- [17] H. Saleur and M. Bauer, Nucl. Phys. **B320** (1989) 591.
- [18] J. Ambjorn, J Jurkiewicz and Y. M. Makeenko, Phys. Lett. **B251** (1990) 517.
- [19] J. Daul, Kazakov and I. Kostov, Nucl. Phys. **B409** (1993) 311  
[arXiv:hep-th/9303093].
- [20] B. Eynard, JHEP **0411:031** (2004) [arXiv:hep-th/0407261].
- [21] M. R. Douglas, Phys. Lett. B **238** (1990) 176; T. Banks, M. Douglas, N. Seiberg  
and S. Shenker, Nucl. Phys. **B238** (1990) 279.
- [22] D. Gross and A. Migdal, Phys. Rev. Lett. **64** (1990) 127; Nucl. Phys. **B340** (1990)  
333.
- [23] M. Anazawa, A. Ishikawa and H. Itoyama, Phys. Rev. **D52** (1995) 6016  
[arXiv:hep-th/9410015]; M. Anazawa and H. Itoyama, Nucl. Phys. **B471** (1996)  
334 [arXiv:hep-th/9511220].